In set theory and logic, **Buchholz's ID hierarchy** is a hierarchy of subsystems of first-order arithmetic. The systems/theories ID_ν are referred to as "the formal theories of ν-times iterated inductive definitions". ID_ν extends PA by ν iterated least fixed points of monotone operators.

Definition

Original definition

The formal theory ID_ω (and ID_ν in general) is an extension of Peano Arithmetic, formulated in the language L_{ID}, by the following axioms:[1]

- $\forall y \forall x(M_y(P_y^M, x) \rightarrow x \in P_y^M)$
- $\forall y(\forall x(M_y(F, x) \rightarrow F(x)) \rightarrow \forall x(x \in P_y^M \rightarrow F(x)))$ for every L_{ID}-formula F(x)
- $\forall y \forall x_0 \forall x_1 ¿¿^M x_0 x_1 \leftrightarrow x_0 < y \wedge x_1 \in P_{x_0}^M ¿$

The theory ID_ν with $\nu \neq \omega$ is defined as:

- $\forall y \forall x(Z_y(P_y^M, x) \rightarrow x \in P_y^M)$
- $\forall x(M_u(F, x) \rightarrow F(x)) \rightarrow \forall x(P_u^M x \rightarrow F(x))$ for every L_{ID}-formula F(x) and each $u < \nu$
- $\forall y \forall x_0 \forall x_1 ¿¿^M x_0 x_1 \leftrightarrow x_0 < y \wedge x_1 \in P_{x_0}^M ¿$

Explanation / alternate definition

ID_1

A set $I \subseteq N$ is called inductively defined if for some monotonic operator $\Gamma: P(N) \rightarrow P(N)$, $LFP(\Gamma) = I$, where $LFP(f)$ denotes the least fixed point of f. The language of ID_1, L_{ID_1}, is obtained from that of first-order number theory, L_N, by the addition of a set (or predicate) constant I_A for every X-positive formula A(X, x) in $L_N[X]$ that only contains X (a new set variable) and x (a number variable) as free variables. The term X-positive means that X only occurs positively in A (X is never on the left of an implication). We allow ourselves a bit of set-theoretic notation:

- $F = \{x \in N / F(x)\}$
- $s \in F$ means $F(s)$

- For two formulas F and G, $F \subseteq G$ means $\forall x F(x) \rightarrow G(x)$.

Then ID_1 contains the axioms of first-order number theory (PA) with the induction scheme extended to the new language as well as these axioms:

- ¿
- ¿

Where $F(x)$ ranges over all L_{ID_1} formulas.

Note that ¿ expresses that I_A is closed under the arithmetically definable set operator $\Gamma_A(S) = \{x \in N / N \vDash A(S, x)\}$, while ¿ expresses that I_A is the least such (at least among sets definable in L_{ID_1}).

Thus, I_A is meant to be the least pre-fixed-point, and hence the least fixed point of the operator Γ_A.

ID_ν

To define the system of ν-times iterated inductive definitions, where ν is an ordinal, let $\prec$ be a primitive recursive well-ordering of order type ν. We use Greek letters to denote elements of the field of $\prec$. The language of ID_ν, L_{ID_ν} is obtained from L_N by the addition of a binary predicate constant J_A for every X-positive $L_N[X, Y]$ formula $A(X, Y, \mu, x)$ that contains at most the shown free variables, where X is again a unary (set) variable, and Y is a fresh binary predicate variable. We write $x \in J_A^\mu$ instead of $J_A(\mu, x)$, thinking of x as a distinguished variable in the latter formula.

The system ID_ν is now obtained from the system of first-order number theory (PA) by expanding the induction scheme to the new language and adding the scheme (TI_ν): $TI(\prec, F)$ expressing transfinite induction along $\prec$ for an arbitrary L_{ID_ν} formula F as well as the axioms:

- ¿
- ¿

where $F(x)$ is an arbitrary L_{ID_ν} formula. In ¿ and ¿ we used the abbreviation $A^\mu(F)$ for the formula $A(F, (\lambda \gamma y; \gamma \prec \mu \wedge y \in J_A^\gamma), \mu, x)$, where x is the distinguished variable. We see that these express that each J_A^μ, for $\mu \prec \nu$, is the least fixed point (among definable sets) for the operator $\Gamma_A^\mu(S) = \{n \in N \vee$ ¿. Note how all the previous sets J_A^γ, for $\gamma \prec \mu$, are used as parameters.

We then define $ID_{\prec \nu} =$ ¿ $\xi \prec \nu ID_\xi$.

Variants

$\dot{ID}_\nu$ - $\dot{ID}_\nu$ is a weakened version of ID_ν. In the system of $\dot{ID}_\nu$, a set $I \subseteq N$ is instead called inductively defined if for some monotonic operator $\Gamma : P(N) \to P(N)$, I is a fixed point of Γ, rather than the least fixed point. This subtle difference makes the system significantly weaker: $PTO(\dot{ID}_1) = \psi(\Omega^{\varepsilon_0})$, while $PTO(ID_1) = \psi(\varepsilon_{\Omega+1})$.

$ID_\nu¿$ is $\dot{ID}_\nu$ weakened even further. In $ID_\nu¿$, not only does it use fixed points rather than least fixed points, and has induction only for positive formulas. This once again subtle difference makes the system even weaker: $PTO(ID_1¿) = \psi(\Omega^\omega)$, while $PTO(\dot{ID}_1) = \psi(\Omega^{\varepsilon_0})$.

$W\text{-}ID_\nu$ is the weakest of all variants of ID_ν, based on W-types. The amount of weakening compared to regular iterated inductive definitions is identical to removing bar induction given a certain subsystem of second-order arithmetic. $PTO(W\text{-}ID_1) = \psi_0(\Omega \times \omega)$, while $PTO(ID_1) = \psi(\varepsilon_{\Omega+1})$.

$U(ID¿¿\nu)¿$ is an "unfolding" strengthening of ID_ν. It is not exactly a first-order arithmetic system, but captures what one can get by predicative reasoning based on ν-times iterated generalized inductive definitions. The amount of increase in strength is identical to the increase from ε_0 to Γ_0 : $PTO(ID_1) = \psi(\varepsilon_{\Omega+1})$, while $PTO¿$.

Results

- Let ν > 0. If a ∈ T_0 contains no symbol D_μ with ν < μ, then "a ∈ W_0" is provable in ID_ν.
- ID_ω is contained in $\Pi^1_1\text{-}CA + BI$.
- If a Π^0_2-sentence $\forall x \exists y \varphi(x, y) (\varphi \in \Sigma^0_1)$ is provable in ID_ν, then there exists $p \in N$ such that $\forall n \geq p \exists k < H_{D_0 D^n_\nu 0}(1) \varphi(n, k)$.
- If the sentence A is provable in ID_ν for all ν < ω, then there exists k ∈ N such that $\vdash^{D^k_\nu 0}_k A^N$.

Proof-theoretic ordinals

- The proof-theoretic ordinal of $\mathrm{ID}_{<\nu}$ is equal to $\psi_0(\Omega_\nu)$.
- The proof-theoretic ordinal of ID_ν is equal to $\psi_0(\varepsilon_{\Omega_\nu+1}) = \psi_0(\Omega_{\nu+1})$.
- The proof-theoretic ordinal of $\mathit{ID}_{¿\omega¿}$ is equal to Γ_0.
- The proof-theoretic ordinal of ID_ν for $\nu<\omega$ is equal to $\varphi(\varphi(\nu,0),0)$.
- The proof-theoretic ordinal of $\mathit{ID}_{\varphi(\alpha,\beta)}$ is equal to $\varphi(1,0,\varphi(\alpha+1,\beta-1))$.
- The proof-theoretic ordinal of $\mathit{ID}_{¿\varphi(0,\alpha)}$ for $\alpha>1$ is equal to $\varphi(1,\alpha,0)$.
- The proof-theoretic ordinal of $\mathit{ID}_{¿\nu¿}$ for $\nu \geq \varepsilon_0$ is equal to $\varphi(1,\nu,0)$.
- The proof-theoretic ordinal of $\mathit{ID}_\nu¿$ is equal to $\varphi(\omega^\nu,0)$.
- The proof-theoretic ordinal of $\mathit{ID}_{¿\nu¿}$ is equal to $\varphi(0,\omega^{\nu+1})$.
- The proof-theoretic ordinal of $W\text{-}\mathit{ID}_{\varphi(\alpha,\beta)}$ is equal to $\psi_0(\Omega_{\varphi(\alpha,\beta)} \times \varphi(\alpha+1,\beta-1))$.
- The proof-theoretic ordinal of $W\text{-}\mathit{ID}_{¿\varphi(\alpha,\beta)}$ is equal to $\psi_0¿$.
- The proof-theoretic ordinal of $U(\mathit{ID}_\nu)$ is equal to $\psi_0(\varphi(\nu,0,\Omega+1))$.
- The proof-theoretic ordinal of $U¿¿¿$ is equal to $\psi_0(\Omega^{\Omega+\varphi(\nu,0,\Omega)})$.
- The proof-theoretic ordinal of ID_1 (the Bachmann-Howard ordinal) is also the proof-theoretic ordinal of KP, $KP\omega$, CZF and $ML_1\ V$.
- The proof-theoretic ordinal of W-ID_ω ($\psi_0(\Omega_\omega\varepsilon_0)$) is also the proof-theoretic ordinal of $W-KPI$.
- The proof-theoretic ordinal of ID_ω (the Takeuti-Feferman-Buchholz ordinal) is also the proof-theoretic ordinal of KPl, $\Pi^1_1-CA+Bl$ and $\Delta^1_2-CA+Bl$.
- The proof-theoretic ordinal of $\mathrm{ID}_{<\omega\hat{}\omega}$ ($\psi_0(\Omega_{\omega^\omega})$) is also the proof-theoretic ordinal of Δ^1_2-CR.
- The proof-theoretic ordinal of $\mathrm{ID}_{<\varepsilon 0}$ ($\psi_0(\Omega_{\varepsilon_0})$) is also the proof-theoretic ordinal of Δ^1_2-CA and Σ^1_2-AC.